

A naturally error suppressing quantum memory

Fumiko Yamaguchi^{1,*} and Yoshihisa Yamamoto^{1,2}

¹*E. L. Ginzton Laboratory, Stanford University, Stanford, CA 94305, USA*

²*National Institute of Informatics, Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan*

(Dated: February 1, 2008)

We propose a method to construct quantum storage wherein the phase error due to decoherence is naturally suppressed without constant error detection and correction. As an example, we describe a quantum memory made of two physical qubits encoded in the ground state of a two-qubit phase-error detecting code. Such a system can be simulated by introducing a coupling between the two physical qubits. This method is effective for physical systems in which the T_1 decay process is negligible but coherence is limited by the T_2 decay process. We take trapped ions as a possible example to apply the natural suppression method and show that the T_2 decay time due to slow ambient fluctuating fields at the physical qubits can be lengthened as much as 10^4 .

PACS numbers: 03.67.Pp, 03.67.Lx, 03.67.Hk

Decoherence in quantum bits (qubits), even if causing only a single qubit error, collapses an exponentially large amount of data and jeopardizes the potential power of quantum computation [1, 2]. Such decoherence also limits the long-lived quantum storage necessary for long-distance quantum communication and distributed quantum computation [3, 4, 5, 6].

Quantum error correction [7, 8, 9, 10] may allow us to regain information from a collapsed state even in the presence of such errors. By fault-tolerant quantum computation, the concept introduced in [11], arbitrarily long quantum computation can be performed even with imperfect logic gates, under the assumption that the error per quantum gate and per qubit during a logic gate is below a threshold value [12, 13, 14, 15, 16]. However, the resource overhead for fault-tolerant error correction is likely to be impractical to implement in an actual physical system.

Storing quantum information in a decoherence-free subspace (DFS) [17, 18, 19, 20] is an excellent way to reduce this overhead. Instead of active error detection and correction using numerous ancilla qubits, which would have been fatal to a conventional fault-tolerant error correction method, the technique uses symmetry of the system such that in the subspace certain causes of decoherence disappear.

One alternative method is natural error suppression by energy consideration [16, 21, 22]. The method incorporates qubit states encoded so that any error that collapses an encoded qubit state costs energy. Unless that amount of energy is supplied by the environment, such an error is suppressed. As in the DFS method, this natural error suppression method eliminates the need for frequent measurement and logic operation to correct quantum errors. This also improves a possible weakness of the DFS method – the coupling of qubits to the environment does not need to possess a certain symmetry required by the

DFS method. In addition, if we design a physical system whose ground state is the code subspace of a well-studied error correcting code, for which fault-tolerant logic operations are known, the logic operations on the encoded qubits are exactly the same as the ones for the error correcting code.

Natural error suppression and correction have been discussed in the context of errors caused by a thermal reservoir [21, 22], and therefore the condition derived for the method to work is that the energy cost of an error must be greater than the thermal energy. However, in many physical systems, this condition is difficult to satisfy and thus natural error suppression may be ineffective. In this paper, we propose a natural error suppression method to reduce decoherence resulting from low-frequency field-fluctuation noise. The condition for this natural error suppression is derived by comparison of the energy cost and the cut-off frequency of the field-fluctuation noise. Even a small energy cost would decrease the effect of slow field-fluctuation noise. Trapped ions serve as a good example of the effectiveness of the method, since the main source of decoherence in quantum memories is slow ambient magnetic-field fluctuation, and any decoherence due to coupling to a thermal reservoir is negligible [23, 24, 25, 26, 27, 28, 29]. In such systems, the T_1 decay time is extremely long and bit-flip errors are unlikely to occur. Therefore, we consider constructing an error suppressing memory based on a two-qubit phase-flip error detecting code, and illustrate how much the T_2 decay time can be lengthened by this method. This will provide a method to design quantum memories with reduced decoherence, instead of – or in addition to – other methods such as applying spin echo technique [30, 31, 32] and using magnetic-field-independent transitions [33].

When a logical qubit is encoded as $|0\rangle_L = |00\rangle$ and $|1\rangle_L = |11\rangle$ using two physical qubits, at most one bit-flip error (σ_{1x} , or σ_{2x}) can be detected. Here we denote our physical qubit basis states as $|0\rangle$ and $|1\rangle$, and the associated Pauli matrices as σ_{iq} ($q = x, y, z$) for qubit i ($i = 1, 2$). The code subspace $\{|0\rangle_L, |1\rangle_L\}$ is a set of simultaneous eigenstates of the stabilizer generator $g =$

*Electronic address: yamaguchi@stanford.edu

$\sigma_{1z}\sigma_{2z}$ with eigenvalue +1. A bit-flip error removes the encoded state from the code subspace, and the stabilizer generator has eigenvalue -1. This code can be converted into a phase-flip detecting code (detecting σ_{1z} , or σ_{2z}) by encoding a logical qubit as $|0\rangle_L = |++\rangle$ and $|1\rangle_L = |--\rangle$ in the rotated basis states, $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. The stabilizer generator for this code is $g = \sigma_{1x}\sigma_{2x}$. We use this two-qubit phase-flip detecting code to design a naturally error suppressing quantum memory.

Associated with the two-qubit phase-flip detecting code, we consider a physical system with an interaction between qubits 1 and 2 in the form of

$$H_{\text{ES}} = -2JI_{1x}I_{2x}, \quad (1)$$

where the qubits are two spins $I_{1x} = \frac{1}{2}\sigma_{1x}$, $I_{2x} = \frac{1}{2}\sigma_{2x}$. The ground states comprise the code subspace of the two-qubit phase-flip error detecting code, and excited states correspond to states collapsed from the encoded state due to a phase-flip error. Therefore, unless the energy difference J is supplied from the environment, a phase-flip error is suppressed automatically without the need for measurements to detect the error or logic operations to correct it.

The system under consideration consists of trapped ions, the two hyperfine ground states of which are the physical qubit states. A quantum memory is composed of two such qubits, described by the spin 1/2 operators,

$$H_0 = \omega_0(I_{1z} + I_{2z}), \quad (2)$$

where ω_0 is the hyperfine splitting, and interaction of the qubits with the fluctuating field $H_{iq}(t)$ at qubit i ($i = 1, 2$) in the q -direction ($q = x, y, z$),

$$H_1(t) = \gamma \sum_{i,q} H_{iq}(t) I_{iq}, \quad (3)$$

where γ is the gyro-magnetic ratio, in addition to the error suppression Hamiltonian (1). Here we assume that the hyperfine splitting term H_0 is much larger than the other terms, H_{ES} and $H_1(t)$. In ion trap experiments, the

hyperfine splitting is about 10GHz and the amplitude of magnetic field fluctuation is about 2kHz [24, 27]. Then the equation of motion of the density matrix [34]

$$\frac{d\rho(t)}{dt} = i[\rho(t), H_0 + H_1(t) + H_{\text{ES}}] \quad (4)$$

will lead to the equation of motion in the interaction picture in H_0 , denoted with $*$,

$$\frac{d\rho^*(t)}{dt} = i[\rho^*(t), H_1^*(t) + H_{\text{ES}}^*(t)], \quad (5)$$

where $\rho^*(t) = e^{iH_0 t} \rho(t) e^{-iH_0 t}$, $H_1^*(t) = e^{iH_0 t} H_1(t) e^{-iH_0 t}$ and so on. We further assume that the fluctuating field amplitude is smaller than the interaction between the two qubits and obtain the second-order perturbation expansion,

$$\frac{d\tilde{\rho}^*(t)}{dt} = i[\tilde{\rho}^*(0), \tilde{H}_1^*(t)] - \int_0^t [[\tilde{\rho}^*(0), \tilde{H}_1^*(t-\tau)], \tilde{H}_1^*(t)] d\tau, \quad (6)$$

where $\tilde{\rho}^* = e^{iH_{\text{ES}}^* t} \rho^* e^{-iH_{\text{ES}}^* t} = e^{iH_0 t} e^{iH_{\text{ES}} t} \rho e^{-iH_{\text{ES}} t} e^{-iH_0 t}$ and $\tilde{H}_1^*(t) = e^{iH_{\text{ES}}^* t} H_1^*(t) e^{-iH_{\text{ES}}^* t}$.

To compute the second term in the right-hand side of Eq. (6), we note that

$$\tilde{H}_1^*(t) = e^{iH_0 t} e^{iH_{\text{ES}} t} H_1(t) e^{-iH_{\text{ES}} t} e^{-iH_0 t}, \quad (7)$$

and

$$e^{iH_{\text{ES}} t} H_1(t) e^{-iH_{\text{ES}} t} = \gamma \sum_{iq} H_{iq} K_{iq}(t), \quad (8)$$

where

$$\begin{aligned} K_{ix}(t) &= I_{ix}, \\ K_{iy}(t) &= I_{iy} \cos Jt + 2I_{iz}I_{ix} \sin Jt, \\ K_{iz}(t) &= I_{iz} \cos Jt - 2I_{iy}I_{ix} \sin Jt. \end{aligned} \quad (9)$$

Here $\bar{i} = 2$ and 1 for $i = 1$ and 2 , respectively. The equation of motion of a matrix element $\tilde{\rho}_{\alpha\alpha'}$ is then

$$\begin{aligned} \frac{d\tilde{\rho}_{\alpha\alpha'}^*(t)}{dt} &= \gamma^2 \sum_{\beta, \beta', i, j, q} \int_0^t d\tau e^{i(E_\alpha - E_\beta + E_{\beta'} - E_{\alpha'})t} \langle \alpha | K_{iq}(t-\tau) | \beta \rangle \tilde{\rho}_{\beta\beta'}^*(0) \langle \beta' | K_{jq}(t) | \alpha' \rangle \\ &\quad \times [e^{-i(E_\alpha - E_\beta)\tau} + e^{-i(E_{\beta'} - E_{\alpha'})\tau}] \overline{H_{iq}(t-\tau) H_{jq}(t)} \\ &\quad - \gamma^2 \sum_{\beta, \beta', i, j, q} \int_0^t d\tau [e^{i(E_\beta - E_{\alpha'})t} \tilde{\rho}_{\alpha\beta}^*(0) \langle \beta | K_{iq}(t-\tau) | \beta' \rangle \langle \beta' | K_{jq}(t) | \alpha' \rangle \\ &\quad + e^{i(E_\alpha - E_{\beta'})t} \langle \alpha | K_{iq}(t) | \beta \rangle \langle \beta | K_{jq}(t-\tau) | \beta' \rangle \tilde{\rho}_{\beta'\alpha'}^*(0)] e^{-i(E_\beta - E_{\beta'})\tau} \overline{H_{iq}(t-\tau) H_{jq}(t)}, \end{aligned} \quad (10)$$

where $|\alpha\rangle$ ($\alpha = \pm 1/2$) is an eigenstate of H_0 satisfying $H_0|\alpha\rangle = E_\alpha|\alpha\rangle$ ($E_\alpha = \omega_0\alpha$). Furthermore, we took the

time average of fluctuating field and assumed the time

average of the fluctuating field is zero

$$\overline{H_{iq}(t)} = 0, \quad (11)$$

and the fluctuations of the three components of the field are independent,

$$\overline{H_{iq}(t - \tau)H_{jq'}(t)} = 0, \quad \text{for } q \neq q'. \quad (12)$$

We further simplify Eq. (10) by ignoring terms oscillating at $\pm\omega_0 t$ or faster, and assuming the correlation time of τ_0 of the spectral densities of the fluctuating fields are longer than $1/J$. The assumption allows us to drop the terms oscillating at $\pm 2Jt$, and then we find the equation of motion for $\langle I_{ix} \rangle = \text{Tr}(\rho I_{ix})$,

$$\begin{aligned} \frac{d\langle I_{ix} \rangle}{dt} = & i \sum_{\alpha, \alpha'} [\rho, H_0 + H_{\text{ES}}]_{\alpha, \alpha'} \langle \alpha' | I_{ix} | \alpha \rangle \\ & - \langle I_{ix} \rangle \gamma^2 \left[\frac{k_{yy}(\omega_0 + J) + k_{yy}(\omega_0 - J)}{2} + k_{zz}(J) \right], \end{aligned} \quad (13)$$

where the spectral densities of the fluctuating fields are defined as

$$k_{qq}^{ij}(\omega) = \frac{1}{2} \int_{-\infty}^{+\infty} \overline{H_{iq}(t - \tau)H_{jq}(t)} e^{-i\omega\tau} d\tau. \quad (14)$$

We have assumed the time average is independent of t and zero when τ is greater than a certain critical value. Therefore, decay time of $\langle I_{ix} \rangle$ is

$$\frac{1}{T_{2x}} = \gamma^2 \left[\frac{k_{yy}(\omega_0 + J) + k_{yy}(\omega_0 - J)}{2} + k_{zz}(J) \right]. \quad (15)$$

Note that only the fluctuating field component at qubit 1 k_{yy}^{11} remains in Eq. (15). In the following, we assume that $k_{qq}^{11} = k_{qq}^{22}(\omega) \equiv k_{qq}(\omega)$ for simplicity. Similarly, decay time T_{2y} for $\langle I_{iy} \rangle$ and T_1 for $\langle I_{iz} \rangle$ are obtained as

$$\begin{aligned} \frac{1}{T_{2y}} &= \gamma^2 [k_{xx}(\omega_0) + k_{zz}(J)], \\ \frac{1}{T_1} &= \gamma^2 \left[k_{xx}(\omega_0) + \frac{k_{yy}(\omega_0 + J) + k_{yy}(\omega_0 - J)}{2} \right]. \end{aligned} \quad (16)$$

The T_2 process averaged over precession in the transverse direction is

$$\frac{1}{T_2} = \frac{1}{2} \left(\frac{1}{T_{2x}} + \frac{1}{T_{2y}} \right) = \frac{1}{2T_1} + \gamma^2 k_{zz}(J). \quad (18)$$

These results should be compared to the decay times T_1^0 and T_2^0 in the case where there is no error suppression Hamiltonian,

$$\frac{1}{T_1^0} = \gamma^2 [k_{xx}(\omega_0) + k_{yy}(\omega_0)], \quad \frac{1}{T_2^0} = \frac{1}{2T_1^0} + \gamma^2 k_{zz}(0). \quad (19)$$

Enhanced T_2	Natural error suppression	Active error correction
T_2/T_2^0	$J\tau_0$	$T_2^0/\Delta t$
10^2	10	3×10^2
10^4	10^2	3×10^4
10^6	10^3	3×10^6

TABLE I: The enhanced T_2 decay time by natural error suppression as a function of the coupling strength J between two qubits and the correlation time τ_0 of the spectral densities of fluctuating fields at the qubits. $T_2^0/\Delta t$ in the rightmost column indicates the number of times per decay time T_2^0 of error correction required to achieve the same enhanced T_2 .

We now assume a simple exponential correlation function for the fluctuating field with a correlation time τ_0 . Then the spectral densities are

$$k_{qq}(\omega) = \overline{H_q^2} \frac{\tau_0}{1 + \omega^2 \tau_0^2}. \quad (20)$$

In a physical system where T_1^0 is extremely long, we approximate $1/T_1^0$ as zero in evaluating the effect of the error suppression Hamiltonian on T_2 decay time. With this approximation, $1/T_1$ is also zero since $k_{yy}(\omega_0 \pm J) \simeq k_{yy}(\omega_0)$ for $J \ll \omega_0$. The T_2 decay time is then enhanced by

$$\frac{T_2}{T_2^0} = \frac{k_{zz}(0)}{k_{zz}(J)} = 1 + (J\tau_0)^2 \quad (21)$$

due to the error suppression Hamiltonian. In the case of trapped ions, J can be made on the order of 10kHz [35] and the dominant noise component is at 50Hz [27]. Assuming that this noise component is the maximum frequency component, we obtain an enhancement of T_2 that is on the order of 10^4 by the error suppression Hamiltonian.

Now we compare the current method to the conventional error correction scheme. We consider a phase error due to the T_2 process at the rate $\epsilon = 1 - e^{-\Delta t/T_2^0}$, in which we perform error detection and correction at every time interval Δt , using the three-qubit phase-flip code [20]. After the error correction, the error rate is reduced to $3\epsilon^2$, corresponding to a lengthened effective decay time T_2^{eff} defined by

$$3\epsilon^2 = 1 - e^{-\Delta t/T_2^{\text{eff}}}. \quad (22)$$

The T_2 increases as $(J\tau_0)^2$ for $J\tau_0 \gg 1$ by natural error suppression, while T_2^{eff} does linearly with the number of error corrections per T_2^0 ($T_2^0/\Delta t$). As summarized in Table I, the natural error suppression method only requires simulating the static coupling between qubits without the need for measurements or logic operations, when the coupling can be made large enough. In contrast, the conventional error correcting method requires a large number of error detections and corrections, which involve measurements and logic operations, to achieve the same enhanced T_2 decay time.

Thus far, we have discussed the enhancement of T_2 by natural error suppression for a stored qubit. Our storage method uses the basis state $|0\rangle_L = |++\rangle$ and $|1\rangle_L = |--\rangle$ for encoding. Suppose we have qubit information in the $|0\rangle/|1\rangle$ basis, $c_0|0\rangle + c_1|1\rangle$, and encode it in our storage. To do so, we first transfer the information to qubit 1 and apply the Hadamard transformation. Qubit 2 is initialized as $|+\rangle$. A controlled-NOT gate between qubits 1 and 2, conditioned on qubit 1, will lead the two-qubit state in $c_0|0\rangle_L + c_1|1\rangle_L$. The decoding process is exactly the reverse of the encoding process; we first apply a controlled-NOT gate, conditioned on qubit 1, and then apply the Hadamard transformation on qubit 1. Then the information is stored in qubit 1 as $c_0|0\rangle + c_1|1\rangle$.

We have shown that the decay time T_2 can be naturally lengthened by simulating a coupling between physical qubits that constitute an encoded qubit. We discussed the case where the main source for the T_2 process is fluctuation of precession frequencies of the qubits in the transverse direction with no energy dissipation (described by the term $\gamma \sum_i H_{iz} I_{iz}$), driven by the zero-frequency fluctuating field. Once the error suppression Hamiltonian is introduced, the fluctuation of precession frequencies requires a finite energy, driven by the higher-frequency fluctuation field. When such an energy change is larger than the bandwidth of the spectral densities of the fluctuating field, the fluctuation of the precession frequencies are not allowed and therefore the T_2 process is suppressed. The condition for natural error suppression to be effective is determined by comparison between the coupling strength between physical qubits and the cut-

off frequency of the ambient fluctuating field. For slowly fluctuating field, even a small coupling between physical qubits is effective in reducing decoherence. Furthermore, unlike the DFS method, there is no required symmetry in the qubit-reservoir coupling.

In this paper, we have shown the benefits of natural error suppression by the simplest possible example – a quantum memory using the two-qubit phase error detecting code. Our proposed method can easily extend to a quantum memory using three physical qubits, whose ground state is the code subspace of the three-qubit phase-flip error correcting code $\{|+++\rangle, |--\rangle\}$. The stabilizer of the code consists of $\sigma_{1x}\sigma_{2x}$ and $\sigma_{2x}\sigma_{3x}$ and the error suppression Hamiltonian can be constructed solely with two-body interactions, $-J(\sigma_{1x}\sigma_{2x} + \sigma_{2x}\sigma_{3x})$. With this addition of an extra physical qubit, the conventional error correction, with the help of ancilla qubits [20] or robust probe modes [36], can be applied to further decrease the error rate when natural error suppression takes place, or we can enjoy the automatic error correcting property even when errors occur [21]. The method can also be extended to suppress not only storage errors but also gate errors for universal quantum computation, using an error-detecting code involving more physical qubits [22]. This will provide an alternative approach to the DFS method, and alleviate the overhead of conventional quantum error correction and fault-tolerant quantum computation.

Acknowledgments: This work was supported in part by JST SORST and NTT Basic Research Laboratories.

-
- [1] L. K. Grover, Phys. Rev. Lett. **79**, 325 (1997).
 - [2] P. W. Shor, in *Proceedings 35th Annual Symposium on Foundations of Computer Science* (IEEE Comput. Soc. Press, 1994), p. 124.
 - [3] C. H. Bennett and G. Brassard, in *Proceedings of IEEE International Conference on Computer Systems and Signal Processing* (1984), vol. 82, p. 175.
 - [4] L. K. Grover, quant-ph/9704012 (1997).
 - [5] H. J. Briegel, W. Dur, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **81**, 5932 (1998).
 - [6] C. H. Bennett *et al.*, Phys. Rev. Lett. **76**, 722 (1996).
 - [7] P. W. Shor, Phys. Rev. A **52**, R2493 (1995).
 - [8] A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996).
 - [9] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).
 - [10] A. Steane, Proc. Roy. Soc. London Ser. A **452**, 2551 (1996).
 - [11] P. W. Shor, in *Proceedings of 37th Annual Symposium on Foundations of Computer Science* (IEEE Comput. Soc. Press, 1996), p. 56.
 - [12] A. Kitaev, in *Quantum Communication, Computing and Measurement, Proceedings of the 3rd International Conference of Quantum Communication and Measurement*, edited by O. Hirota, A. S. Holevo, and C. M. Caves (Plenum Press, New York, 1997), p. 181.
 - [13] J. Preskill, Proc. R. Soc. London, Ser. A **454**, 385 (1998).
 - [14] E. Knill, R. Laflamme, and W. H. Zurek, Science **279**, 342 (1998).
 - [15] D. Aharonov and M. Ben-Or, in *Proceedings of the 29th Annual ACM Symposium on Theory of Computing*, edited by F. T. Leighton and P. Shor (ACM, New York, 1998), p. 176.
 - [16] A. Kitaev, quant-ph/9707021 (1997).
 - [17] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).
 - [18] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).
 - [19] L.-M. Duan and G.-C. Guo, Phys. Rev. A **57**, 737 (1998).
 - [20] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).
 - [21] J. P. Barnes and W. S. Warren, Phys. Rev. Lett. **85**, 856 (2000).
 - [22] D. Bacon, K. R. Brown, and K. B. Whaley, Phys. Rev. Lett. **10**, 247902 (2001).
 - [23] P. T. H. Fisk *et al.*, IEEE Transactions on Instrumentation and Measurement **44**, 113 (1995).
 - [24] D. J. Wineland *et al.*, J. Res. Natl. Inst. Stand. Technol. **103**, 259 (1998).
 - [25] D. Kielpinski, C. Monroe, and D. J. Wineland, Nature

- 417**, 709 (2002).
- [26] M. A. Rowe *et al.*, Quantum Information and Computation **2**, 257 (2002).
 - [27] F. Schmidt-Kaler *et al.*, J. Phys. B **36**, 623 (2003).
 - [28] D. J. Wineland *et al.*, Phil. Trans. R. Soc. Lond A **361**, 1349 (2003).
 - [29] J. Chiaverini *et al.*, Science **13**, 997 (2005).
 - [30] M. F. Andersen, A. Kaplan, and N. Davidson, Phys. Rev. Lett. **90** (2003).
 - [31] M. Riebe *et al.*, **429**, 734 (2004).
 - [32] M. D. Barrett *et al.*, **429**, 737 (2004).
 - [33] C. Langer *et al.*, Phys. Rev. Lett. **95**, 060502 (2005).
 - [34] C. P. Slichter, *Principles of Magnetic Resonance* (Springer, 1996), chap. 5.12, p. 206, 3rd ed.
 - [35] D. Porras and J. I. Cirac, Phys. Rev. Lett. **92**, 207901 (2004).
 - [36] F. Yamaguchi, K. Nemoto, and W. J. Munro, quant-ph/0511098 (2005).